

# NON-REALIZABLE MINIMAL VERTEX TRIANGULATIONS OF SURFACES: SHOWING NON-REALIZABILITY USING ORIENTED MATROIDS AND SATISFIABILITY SOLVERS

LARS SCHEWE

**ABSTRACT.** We show that no minimal vertex triangulation of a closed, connected, orientable 2-manifold of genus 6 admits a polyhedral embedding in  $\mathbb{R}^3$ . We also provide examples of minimal vertex triangulations of closed, connected, orientable 2-manifolds of genus 5 that do not admit any polyhedral embeddings. We construct a new infinite family of non-realizable triangulations of surfaces. These results were achieved by transforming the problem of finding suitable oriented matroids into a satisfiability problem. This method can be applied to other geometric realizability problems, e.g. for face lattices of polytopes.

Grünbaum conjectured [15, Exercise 13.2.3] that all triangulated surfaces (compact, orientable, connected, 2-dimensional manifolds without boundary) admit polyhedral embeddings in  $\mathbb{R}^3$ . This conjecture was shown to be false by Bokowski and Guedes de Oliveira [5]. They showed that one special triangulation with 12 vertices of a surface of genus 6 does not admit a polyhedral embedding in  $\mathbb{R}^3$ . Recently, Archdeacon et al. [2] settled the case of genus 1 by showing that all triangulations of the torus admit a polyhedral embedding.

Still, triangulated surfaces with polyhedral embeddings can be quite complicated. McMullen, Schulz, and Wills constructed polyhedral embeddings of triangulated surfaces with  $n$  vertices of genus  $\Theta(n \log n)$  ([23], see also [30]). However, a gap remains: Jungerman and Ringel [18, 27] showed that  $n$  vertices suffice to triangulate a surface of genus  $\Theta(n^2)$  and explicitly constructed such triangulations.

So, can we construct polyhedral embeddings of triangulated surfaces with few vertices? In the case of 2-spheres the combinatorial bound is sharp; this is a consequence of Steinitz's Theorem [28]. It is known that all vertex minimal triangulations of surfaces up to genus 4 admit polyhedral embeddings (genus 1 was first done by Császár [11], the

---

2000 *Mathematics Subject Classification.* Primary 52B70; Secondary 52C40.

The author was supported by a scholarship of the Deutsche Telekom Foundation.

cases of genus 2 and 3 were solved by Lutz and Bokowski [19], Lutz [19] and Hougardy, Lutz, and Zelke [17]).

Our main result is that none of the vertex minimal triangulations of a surface of genus 6 admits a realization in  $\mathbb{R}^3$ . Moreover, three minimal triangulations of a surface of genus 5 do not admit realizations either. A small modification of one of triangulations help us to construct a new infinite class of non-realizable triangulated surfaces. For all results we use an improved method to construct oriented matroids that are admissible for the surface in question. The method can also be applied to embedding problems for general simplicial complexes in arbitrary dimensions. A small modification of the method allows us to also treat immersions of simplicial complexes. Using this modification we can rule out for all but one triangulation of the surface of genus 6 with 12 vertices that it can be immersed into  $\mathbb{R}^3$ .

The new method we propose to generate oriented matroids reduces the generation problem to an instance of the satisfiability problem. This allows us to use well-tuned software and speeds up the checking process immensely. As oriented matroids have been used to tackle other geometric realizability problems, our method gives more effective algorithms for these problems as well.

## 1. RESULTS

Using the algorithm given below, it was possible to show the following theorems:

**Theorem 1.1.** *No triangulation of a surface of genus 6 with 12 vertices admits a polyhedral realization in  $\mathbb{R}^3$ .*

The theorem is a consequence of the following proposition. A key step is the classification of combinatorial surfaces with 12 vertices of genus 6 by Altshuler, Bokowski and Schuchert [1].

**Proposition 1.2.** *None of the 59 combinatorial surfaces with 12 vertices of genus 6 admits an acyclic, uniform oriented matroid.*

The situation is more difficult in the case of genus 5. To triangulate a surface of genus 5 we also need at least 12 vertices. However, there are far more possibilities (751 593 as enumerated by Lutz and Sulanke [21]) than in the case of genus 6.

Nevertheless, the next theorem shows that the case of genus 5 looks also more interesting.

**Theorem 1.3.** *There exist at least three combinatorially distinct triangulations of a surface of genus 5 with 12 vertices that do not admit*

TABLE 1. Number of combinatorial triangulations

$g$	$n_{\min}$	#
0	4	1
1	7	1
2	10	865
3	10	20
4	11	821
5	12	751 593
6	12	59

 TABLE 2. Triangulation  ${}^212_1^1$  of [20]

1	2	3	1	2	12	1	3	6	1	4	8	1	4	11	1	5	9
1	5	10	1	6	9	1	8	10	1	11	12	2	3	4	2	4	7
2	5	9	2	5	12	2	6	10	2	6	11	2	7	10	2	9	11
3	4	5	3	5	8	3	6	10	3	7	11	3	7	12	3	8	11
3	10	12	4	5	6	4	6	9	4	7	11	4	8	12	4	9	12
5	6	7	5	7	10	5	8	12	6	7	8	6	8	11	7	8	9
7	9	12	8	9	10	9	10	11	10	11	12						

 TABLE 3. Triangulation  ${}^212_2^1$  of [20]

1	2	3	1	2	12	1	3	6	1	4	9	1	4	11	1	5	8
1	5	9	1	6	10	1	8	10	1	11	12	2	3	4	2	4	7
2	5	10	2	5	12	2	6	9	2	6	10	2	7	11	2	9	11
3	4	5	3	5	8	3	6	11	3	7	10	3	7	11	3	8	12
3	10	12	4	5	6	4	6	9	4	7	12	4	8	11	4	8	12
5	6	7	5	7	10	5	9	12	6	7	8	6	8	11	7	8	9
7	9	12	8	9	10	9	10	11	10	11	12						

a polyhedral realization in  $\mathbb{R}^3$ . However, there exists at least one triangulation of a surface of genus 5 with 12 vertices that admits many oriented matroids.

Specifically no admissible oriented matroids exist for the manifolds  ${}^212_1^1$ ,  ${}^212_2^1$ , and  ${}^212_6^1$  described in the dissertation of Frank Lutz [20]. However, more than 100 000 admissible oriented matroids exist for the manifold  ${}^212_1^5$ . A facet description of the non-realizable manifolds can be found in the Tables 2, 3, 4.

Another interesting question was dealt with by Bokowski and Guedes de Oliveira [5]: Are there infinite classes of surfaces of a fixed genus that cannot be realized? Bokowski and Guedes de Oliveira tried to answer

TABLE 4. Triangulation  ${}^212_6^1$  of [20]

1	2	4	1	2	6	1	3	6	1	3	12	1	4	11	1	5	9
1	5	12	1	8	9	1	8	10	1	10	11	2	3	5	2	3	7
2	4	7	2	5	12	2	6	10	2	9	10	2	9	11	2	11	12
3	4	6	3	4	8	3	5	8	3	7	11	3	10	11	3	10	12
4	5	7	4	5	9	4	6	9	4	8	12	4	11	12	5	6	8
5	6	10	5	7	10	6	7	9	6	7	11	6	8	11	7	8	10
7	8	12	7	9	12	8	9	11	9	10	12						

TABLE 5. Surface no.1 of [1]

1	2	11	1	2	12	1	3	4	1	3	10	1	4	9	1	5	6
1	5	11	1	6	9	1	7	8	1	7	10	1	8	12	2	3	6
2	3	8	2	4	10	2	4	12	2	5	9	2	5	10	2	6	7
2	7	11	2	8	9	3	4	6	3	5	7	3	5	12	3	7	8
3	9	10	3	9	11	3	11	12	4	5	8	4	5	12	4	6	8
4	7	10	4	7	11	4	9	11	5	6	10	5	7	9	5	8	11
6	7	12	6	8	9	6	10	11	6	11	12	7	9	12	8	10	11
8	10	12	9	10	12												

this question by taking a non-realizable surface and cutting out a triangle such that the remaining manifold stays non-realizable. We found that the remaining manifold given by Bokowski and Guedes de Oliveira admits chirotopes after all. Their argument for non-realizability depends crucially on the symmetry of the surface to reduce the search space. However, the symmetry group of the manifold in question is smaller than the symmetry group of the whole surface.

Still, our algorithm yields an even stronger statement:

**Theorem 1.4.** *For each genus  $g \geq 5$  there exist infinite classes of surfaces that have no polyhedral embedding in  $\mathbb{R}^3$ .*

The main idea to construct such an infinite family was already given by Bokowski and Guedes de Oliveira [5]. We take the connected sum of suitable surfaces; we can ensure that the result is non-realizable if one of the summands stayed non-realizable after the removal of one triangle. As we do not need to impose any conditions on the second summand, we can then construct surfaces of arbitrary genus  $g$  as long as  $g$  is greater or equal than the genus of the first summand; by additionally adding triangulations of spheres with arbitrary numbers of vertices we can construct the infinite families we are after. The construction is summarized in the following Lemma. We omit the straight-forward proof.

**Lemma 1.5.** *Given two triangulations  $S$  and  $T$  of surfaces and a triangle  $T \in S$  such that  $S \setminus \{T\}$  is non-realizable, then there exists a triangulation  $X$  with  $V(X) = V(S) + V(T) - 3$  vertices of the surface of genus  $g_S + g_T$  that is non-realizable as well.*

The following proposition shows that the conditions of the Lemma can be satisfied.

**Proposition 1.6.** *a) Let  $O$  be the surface  $^212_1^1$  as above and let  $M := O \setminus \{\{1, 2, 3\}\}$ . Then  $M$  does not admit an acyclic uniform oriented matroid.*  
*b) Let  $P$  be the surface no. 1 in the enumeration of Altshuler and Bokowski [1] (see Table 5) and let  $N := P \setminus \{\{1, 2, 11\}\}$ . Then  $N$  does not admit an acyclic uniform oriented matroid.*

*Proof of Theorem 1.4.* As a first step we will show a construction that yields for any surface  $X$  a non-realizable surface  $S$ . We then exhibit suitable sequences of surfaces to show the Theorem.

Take a triangulated surface  $X$  of genus  $g$  with  $n$  vertices. After renumbering we may assume that the vertices are  $13, \dots, n+12$  and that  $[n+10, n+11, n+12]$  is a triangle in  $X$ . Now we take the connected sum of  $X$  and  $O$  where we identify the pairs of vertices  $(1, n+10)$ ,  $(2, n+11)$ , and  $(3, n+12)$ . We call this complex  $S$ . It follows from the construction that  $S$  is a surface. Furthermore,  $S$  has genus  $g+5$  and  $n+9$  vertices. We claim that  $S$  cannot be realizable: it contains  $M$  as a subcomplex. As we have seen  $M$  is not realizable, so the claim follows.

Now, let  $g \geq 5$ . Then let  $X_0$  be any triangulated surface of genus  $g-5$  and let  $X_i$  be the connected sum of  $X_0$  with a triangulated sphere with  $i+3$  points. We see that the sequence  $S_0, S_1, \dots$  constructed as above is an infinite sequence of surfaces of genus  $g$  all of which are not realizable.  $\square$

Our results depend on the following method to generate oriented matroids. We first give an overview before we deal with the technical details.

## 2. THE MAIN ALGORITHM

We want to treat the embeddability problem algorithmically. To do so, we need a combinatorial model of a point set in  $\mathbb{R}^n$ , which captures interesting properties (for instance, convexity). Oriented matroids are a good choice for this purpose. Examples of such applications can be found for instance in the book by Bokowski and Sturmfels [6].

In the realizable case the circuits of an oriented matroid correspond to minimal Radon partitions of the corresponding elements. We can use

this correspondence to check whether two simplices intersect each other. If  $F$  and  $G$  are simplices such that  $F \cap G = \emptyset$ , they intersect if and only if  $F \cup G$  contains a circuit  $C$  such that  $C^+ \subseteq F$  and  $C^- \subseteq G$ . We say that an oriented matroid  $\mathcal{M} = (E, \chi)$  is *admissible for a simplicial complex*  $\mathbf{K}$  if  $E = |\mathbf{K}|$  and for all  $F, G \in \mathbf{K}$  with  $F \cap G = \emptyset$  there does not exist any circuit  $C$  such that  $C^+ \subseteq F$  and  $C^- \subseteq G$ . If we consider only uniform oriented matroids of rank 4 and our simplices are faces of a surface, we only need to consider the case that  $F$  is a triangle and  $G$  is an edge. Additionally, we use a known fact about oriented matroids that are derived from point sets: no circuit of such an oriented matroid is totally positive. Oriented matroids with this property are called *acyclic*.

We can restrict our problem even further: polyhedral embeddings of triangulated surfaces are “nice”; we can perturb the vertices by a small amount without creating any intersections of the triangles. This makes our task of finding oriented matroids comparatively easy. We can restrict our attention to *uniform* oriented matroids.

So, for a given simplicial complex, we can deduce that  $\mathbf{K}$  cannot be embedded in  $\mathbb{R}^d$ , if  $\mathbf{K}$  does not admit any acyclic, uniform oriented matroid of rank  $d + 1$ . We will now check for this condition by transforming it into an instance of SAT. Luckily, this transformation is quite straightforward. However, we first review some oriented matroid terminology and fix the notation for instances of SAT. The main part consists of the encoding the oriented matroid axioms, i.e. the three-term Grassmann-Plücker relations, and of encoding the “forbidden” circuits.

**2.1. Simplicial Complexes.** We now give a rough sketch how oriented matroids can be used to tackle realizability questions. Assume we have a realization of a triangulated surface  $\mathbf{S}$ , i.e. a map  $f : \mathbf{S} \rightarrow \mathbb{R}^3$  such that for all  $\Delta_1, \Delta_2 \in \mathbf{S}$  holds that  $\text{conv}(f(\Delta_1 \cap \Delta_2)) = f(\text{conv}(\Delta_1)) \cap f(\text{conv}(\Delta_2))$ . If we want that  $f$  is an embedding, we need to make sure that the image of two simplices has non-trivial intersection if and only if the simplices themselves intersected non-trivially.

**Definition 2.1** (Embedding). Given a triangulation  $\mathbf{K}$  of a surface, we say that a mapping  $f : \mathbf{K} \rightarrow \mathbb{R}^d$  induces an embedding if for no two simplices that are disjoint in  $\mathbf{K}$  their images under  $f$  intersect in  $\mathbb{R}^d$ .

When we want to check whether a mapping is an embedding, we can restrict our attention to simplices whose dimension sum to  $d$ . In our case this means we only need to check intersections of one triangle with an edge that is disjoint from the triangle.

**2.2. Oriented Matroids.** The following discussion of oriented matroids is extremely brief, we recommend the monograph [3], especially Section 3.5 for the missing details.

We only consider uniform oriented matroids and assume these are given by their chirotopes. We also assume that the ground set  $E$  of the oriented matroids is  $\{1, \dots, n\}$ . We use the following axioms for oriented matroids:

**Definition 2.2.** Let  $E = \{1, \dots, n\}$ ,  $r \in \mathbb{N}$ , and  $\chi : E^r \rightarrow \{-1, +1\}$ . We call  $\mathcal{M} = (E, \chi)$  a uniform oriented matroid of rank  $r$ , if the following conditions are satisfied:

- (B1) The mapping  $\chi$  is alternating.
- (B2) For all  $\sigma \in \binom{n}{r-2}$  and all subsets  $\{x_1, \dots, x_4\} \subseteq E \setminus \sigma$  the following holds:

$$\{\chi(\sigma, x_1, x_2)\chi(\sigma, x_3, x_4), -\chi(\sigma, x_1, x_3)\chi(\sigma, x_2, x_4), \chi(\sigma, x_1, x_4)\chi(\sigma, x_2, x_3)\} \supseteq \{-1, +1\}$$

*Remark 2.3.* The mapping  $\chi$  is called the *chirotope* of the oriented matroid.

As a first consequence of these axioms we can restrict our attention to the values that  $\chi$  attains on the ordered  $r$ -subsets of  $E$ . The other values are then determined by (B1).

The class of oriented matroids we are interested in is still smaller than the class of uniform oriented matroids. We also want our oriented matroids to be *acyclic*, that means they should contain no circuit in which every element has positive signature. Oriented matroids with a positive circuit are called *cyclic*.

Given a uniform oriented matroid  $\mathcal{M} = (E, \chi)$  the circuit signatures of  $\mathcal{M}$  can be computed from the chirotope: Let  $\underline{C} = [c_1, \dots, c_{r+1}]$  ( $c_1 < \dots < c_{r+1}$ ) be the unoriented circuit, then the two possible signatures  $C^+$  and  $C^-$  of  $\underline{C}$  are given by  $C_i = (-1)^i \chi[c_1, \dots, \widehat{c_i}, \dots, c_{r+1}]$  and its negative (for a proof, see [3, Lemma 3.57]). Recalling the discussion in the section above, the circuit signatures give us the possibility to check whether two simplices of complementary dimensions intersect.

**2.3. SAT.** Before we give our transformation, we first fix our notation for instances of SAT.

Take a Boolean function  $\Phi : \{0, 1\}^n \rightarrow \{0, 1\}$ , where 0 stands for false and 1 for true. We call the elements of  $\{0, 1\}^n$  *valuations*. A valuation is *satisfying* if  $\Phi(v) = 1$ .

We transform our problem, whether there exists an admissible oriented matroid for a given simplicial complex, into an instance of SAT.

An instance of SAT consists of a boolean function given in conjunctive normal form (CNF). That is, given the variables  $p_1, \dots, p_n$  the function  $\Phi$  is of the form  $\Phi(p) = \bigwedge_{i=1}^m C_i$  where the  $C_i$  are of the form  $C_i = \bigvee_{j \in I_i} p_j \vee \bigvee_{j \in \bar{I}_i} \bar{p}_j$ . A SAT solver answers the question whether  $\Phi$  is satisfiable. In that case it returns a valuation  $v$  such that  $\Phi(v) = 1$ .

The following observation goes back to Peirce [25]. It gives us a way to write an arbitrary boolean function in CNF.

**Lemma 2.4.** *Let  $\Phi$  be a boolean function  $\Phi : \{0, 1\}^n \rightarrow \{0, 1\}$ . Then we can write  $\Phi$  as:*

$$\Phi(x) = \bigwedge_{\substack{\sigma \in \{0,1\}^n \\ \neg \Phi(\sigma)}} \left( \bigvee_{i \in \{j | \sigma_j = 1\}} \bar{x}_i \right) \vee \left( \bigvee_{i \in \{j | \sigma_j = 0\}} x_i \right)$$

**2.4. Encoding.** We are now ready to give the transformation of our problem: Given a simplicial complex  $K$  on  $n$  points and a dimension  $d$ , we want to decide whether there exists an acyclic, uniform oriented matroid of rank  $d + 1$  on  $n$  points that is admissible for  $K$ .

To encode the chirotope we introduce a variable for each ordered  $r$ -subset  $B$  of  $\{1, \dots, n\}$  which we denote by  $[B]$ . Given a valuation  $v$  we construct a chirotope  $\chi_v$  as follows: If  $v_{[B]} = 1$ , we set  $\chi_v(B) = +1$  and if  $v_{[B]} = 0$  then we set  $\chi_v(B) = -1$ .

We start by encoding the oriented matroid axioms. We do not deal explicitly with the axiom (B1) as we only fix the signs for the ordered subsets. The following proposition allows us to deal with axiom (B2). It follows directly from Lemma 2.4.

**Proposition 2.5.** *Let  $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$  be ordered  $r$ -subsets of  $E$ ,  $v \in \{0, 1\}^{\binom{|E|}{r}}$ , and  $\chi_v$  defined as above. Then the following two conditions are equivalent:*

- (1)  $\{\chi_v(\alpha)\chi_v(\beta), -\chi_v(\gamma)\chi_v(\delta), \chi_v(\epsilon)\chi_v(\zeta)\} \supseteq \{+1, -1\}$
- (2)  $v$  satisfies  $GP(\alpha, \beta, \gamma, \delta, \epsilon, \zeta)$  as defined in Table 6.

So, three-term Grassmann-Plücker relation is encoded with 16 clauses with 6 literals each. As we have  $\binom{n}{r-2} \binom{n-r+2}{4}$  different Grassmann-Plücker relations to consider, we get  $16 \binom{n}{r-2} \binom{n-r+2}{4}$  many clauses of length 6 in our resulting SAT instance. These clauses guarantee the property that each satisfiable valuation of the instance will correspond to a chirotope.

To complete the model we need a condition that excludes all oriented matroids that have a given circuit signature. As a special case we want to exclude all cyclic oriented matroids. The following proposition



$$\begin{aligned}
GP(\alpha, \beta, \gamma, \delta, \epsilon, \zeta) = & (\neg[\alpha] \vee \neg[\beta] \vee \neg[\gamma] \vee [\delta] \vee \neg[\epsilon] \vee \neg[\zeta]) \\
& \wedge (\neg[\alpha] \vee \neg[\beta] \vee \neg[\gamma] \vee [\delta] \vee [\epsilon] \vee [\zeta]) \\
& \wedge (\neg[\alpha] \vee \neg[\beta] \vee [\gamma] \vee \neg[\delta] \vee \neg[\epsilon] \vee \neg[\zeta]) \\
& \wedge (\neg[\alpha] \vee \neg[\beta] \vee [\gamma] \vee \neg[\delta] \vee [\epsilon] \vee [\zeta]) \\
& \wedge (\neg[\alpha] \vee [\beta] \vee \neg[\gamma] \vee \neg[\delta] \vee \neg[\epsilon] \vee [\zeta]) \\
& \wedge (\neg[\alpha] \vee [\beta] \vee \neg[\gamma] \vee \neg[\delta] \vee [\epsilon] \vee \neg[\zeta]) \\
& \wedge (\neg[\alpha] \vee [\beta] \vee [\gamma] \vee [\delta] \vee \neg[\epsilon] \vee [\zeta]) \\
& \wedge (\neg[\alpha] \vee [\beta] \vee [\gamma] \vee [\delta] \vee [\epsilon] \vee \neg[\zeta]) \\
& \wedge ([\alpha] \vee \neg[\beta] \vee \neg[\gamma] \vee \neg[\delta] \vee \neg[\epsilon] \vee [\zeta]) \\
& \wedge ([\alpha] \vee \neg[\beta] \vee \neg[\gamma] \vee \neg[\delta] \vee [\epsilon] \vee \neg[\zeta]) \\
& \wedge ([\alpha] \vee \neg[\beta] \vee [\gamma] \vee [\delta] \vee \neg[\epsilon] \vee [\zeta]) \\
& \wedge ([\alpha] \vee \neg[\beta] \vee [\gamma] \vee [\delta] \vee [\epsilon] \vee \neg[\zeta]) \\
& \wedge ([\alpha] \vee [\beta] \vee \neg[\gamma] \vee [\delta] \vee \neg[\epsilon] \vee \neg[\zeta]) \\
& \wedge ([\alpha] \vee [\beta] \vee \neg[\gamma] \vee [\delta] \vee [\epsilon] \vee [\zeta]) \\
& \wedge ([\alpha] \vee [\beta] \vee [\gamma] \vee \neg[\delta] \vee \neg[\epsilon] \vee \neg[\zeta]) \\
& \wedge ([\alpha] \vee [\beta] \vee [\gamma] \vee \neg[\delta] \vee [\epsilon] \vee [\zeta])
\end{aligned}$$

TABLE 6. Definition of the function  $GP$ 

gives the exact condition; again the proposition follows directly from Lemma 2.4:

**Proposition 2.6.** *Let  $\mathcal{M} = (E, \chi)$  be a uniform oriented matroid of rank  $r$ ,  $v$  the corresponding valuation and  $C = (s_1 c_1, \dots, s_{r+1} c_{r+1})$  be a signed  $(r+1)$ -tuple ( $s_i \in \{+1, -1\}$ ,  $c_i \in \{1, \dots, n\}$ ,  $c_i \neq a_j$ ). Then  $C$  is not a circuit of  $\chi$  if and only if  $v$  satisfies  $\Gamma(C)$ :*

$$\begin{aligned}
\Gamma(C) = & \bigwedge_{i \in I^+} [c_1, \dots, \widehat{c}_i, \dots, c_{r+1}] \wedge \bigwedge_{i \in I^-} \neg[c_1, \dots, \widehat{c}_i, \dots, c_{r+1}] \\
& \vee \bigwedge_{i \in I^+} \neg[c_1, \dots, \widehat{c}_i, \dots, c_{r+1}] \wedge \bigwedge_{i \in I^-} [c_1, \dots, \widehat{c}_i, \dots, c_{r+1}] \\
I^+ = & \{i \mid (-1)^i s_i = +1\} \\
I^- = & \{i \mid (-1)^i s_i = -1\}
\end{aligned}$$

Thus, we add for every forbidden circuit two clauses consisting of  $r + 1$  literals each. With these clauses we have completed our SAT-model. In the next section we will see how this gives us an effective way to solve our problem. If we want to use this method to treat other realizability problems, other restrictions are of interest. In the case of the algorithmic Steinitz problem, i.e. whether a lattice is a face lattice of a convex polytope, we need to generate oriented matroids with prescribed cocircuits. The necessary clauses can be derived in the same manner as described above.

### 3. IMPLEMENTATION

We wrote a Haskell [26] program that does the translation described in the preceding section. We then used the SAT-solvers ZChaff [22] and Minisat [12] to solve the resulting SAT instances. To verify the data entry of the 59 Altshuler examples we checked the resulting surfaces using Polymake [14].

We tested our programs on known examples. We computed all chirotopes that are admissible for the Möbius torus. We found 2772 chirotopes (in less than 20 seconds) which is the same number that Bokowski and Eggert [4] found. Furthermore, we tested all triangulated surfaces with up to 9 vertices (including the non-orientable ones). In that case our program correctly found out which surfaces (all the orientable ones) admitted chirotopes and which did not.

Additionally, we used our program to verify that all 821 minimal vertex triangulations of a surface of genus 4 as classified by Lutz and Sulanke [21] admit a chirotope.

There are quite a number of software packages to generate oriented matroids (for instance [5, 8, 13, 16]). These packages use one of two different approaches: The programs by Bokowski and Guedes de Oliveira and by Finschi construct oriented matroids by using single element extensions, whereas the other programs try to construct the oriented matroids globally by filling in the chirotopes. Our approach is of the second type. We want to mention that David Bremner reported on his transformation of the problem into a 0/1 integer program. However, his benchmark results showed that his backtracking program is faster in the instances he used.

To give an impression of the efficiency of our program, we state some running times: For the genus 6 examples the transformation took approximately 30 seconds per instance. Solving the SAT instances took between 22 and 98 minutes. All times were taken on a machine with two Pentium III processors (1 GHz) and 2 GB RAM. For all

computations only one processor was used. These results show that our program is much faster than the program of [5]. We think the most interesting comparison would be to the program MPC of David Bremner. However, he does not implement the possibility to exclude oriented matroids with certain circuits.

One of the advantages of our method lies in the fact that one can use a variety of SAT solvers to check the results. The transformation is simple enough to be checked by hand. Many SAT solvers allow the possibility to give a “proof” that an instance is unsatisfiable. They output how to derive a contradiction from the given input. However, this does not improve our situation: the proofs generated this way are so large that they can only be checked with the help of a computer. Advances in the development of proof assistants might make it possible to give a full formal verification of our results in the near future.

#### 4. IMMERSIONS

We have seen that we cannot hope to find embeddings for all triangulations of orientable surfaces. However, one could hope for weaker results. In the context of non-orientable surfaces, where embeddings cannot be found for topological reasons, one tries instead to find immersions of these surfaces. Thus, we could hope to find some immersions for the surfaces we found not to be embeddable.

We mention that Cervone [10] showed there are non-immersable triangulations with eight vertices of the Klein bottle, whereas one can find an immersion of a triangulation with nine vertices. Brehm had earlier shown that there is no gap between the necessary vertex numbers for immersions of the real projective plane [7].

**Definition 4.1** (Immersion). Given a triangulation  $\mathbf{K}$  of a surface, we say that a mapping  $f : \mathbf{K} \rightarrow \mathbb{R}^d$  induces an immersion if for no two triangles in the star of a vertex  $v \in \mathbf{K}$  their images under  $f$  intersect in  $\mathbb{R}^d$ .

*Remark 4.2.* The star of a vertex is the smallest simplicial complex, that contains all faces that contain the given vertex.

This definition directly leads to an adaptation of the notion of an admissible oriented matroid. We say that an oriented matroid  $\mathcal{M} = (E, \chi)$  is *admissible with respect to an immersion of a simplicial complex  $\mathbf{K}$*  if the following conditions hold:

- $E = |\mathbf{K}|$ ,
- for all  $F, G \in \text{star}(v)$  with  $F \cap G = \emptyset$  there does not exist any circuit  $C$  such that  $C^+ \subseteq F$  and  $C^- \subseteq G$ .

Using a suitably modified version of the algorithm above (one just needs to test fewer possible intersections), we can show that all but one of the 59 surfaces of genus 6 do not admit an oriented matroid that is admissible with respect to an immersion of that surface. The exception is surface number 15 (again using the numbering scheme used by Altshuler et al. [1]).

## 5. CONCLUSION

Our results give additional insight in the properties of minimal vertex triangulations of surfaces. Still, the main problems remain: How can we characterize non-realizability? Are all triangulated surfaces of small genus (i.e.  $g \leq 4$ ) realizable?

The infinite class of non-realizable surfaces given above hints that there will be no easy answer to the first question. For genus 5 and 6 we can construct non-realizable triangulations for any number of vertices. We conjecture that this holds also for any genus larger than 6. However, we think it should be possible to prove that for every genus greater than 4 we need strictly more vertices for a polyhedral embedding of a surface than for a combinatorial triangulation. However, one of the main obstacles for such an investigation is the lack of good construction methods for “interesting” combinatorial surfaces.

The method we used is interesting in its own right. It helps tremendously in the study of small examples. However, we hope that the small examples given here will help in the solution of the general problem. One point that needs improvement is the fact that we cannot use effectively use the information we gain if we find oriented matroids in the course of our search. The methods for finding realizations of oriented matroids are not good enough to yield practical results.

As an open problem remains the question how strong oriented matroid methods are compared to the methods described by Novik [24] and Timmreck [29]. We conjecture that using oriented matroids will give as strong results as the method proposed by Timmreck. We are lead to this conjecture by the result of Carvalho and Guedes de Oliveira [9]. They showed that the linking number arguments given by Brehm as incorporated by Timmreck hold also in the setting of oriented matroids. That means that these arguments are subsumed by the oriented matroid technique.

The technique used in this article can be applied to other geometric problems. It has already been used to treat realizability of point-line

configurations. Another application could be in tackling the Algorithmic Steinitz problem (cf. [3]). We hope that this technique proves itself to be a useful building block in these and other applications.

#### ACKNOWLEDGEMENTS

I would like to thank Jürgen Bokowski for all the advice and encouragement received while undertaking this work. I had a number of helpful discussions with a number of people: I like to mention Peter Lietz, Frank Lutz (who also generously provided me lots of interesting examples for testing my software), and Dagmar Timmreck. Thanks also to Michael Joswig who gave a number of valuable comments on an earlier draft of this article.

#### REFERENCES

- [1] Amos Altshuler, Jürgen Bokowski, and Peter Schuchert, *Neighborly 2-manifolds with 12 vertices*, J. Combin. Theory Ser. A **75** (1996), no. 1, 148–162.
- [2] Dan Archdeacon, C. Paul Bonnington, and Joanna A. Ellis-Monaghan, *How to exhibit toroidal maps in space*, Discrete Comput. Geom. **38** (2007), no. 3, 573–594.
- [3] Anders Björner, Michel Las Vergnas, Bernd Sturmfels, Neil White, and Günter M. Ziegler, *Oriented matroids*, 2nd ed., Encyclopedia of Mathematics and its Applications, vol. 46, Cambridge University Press, Cambridge, 1999.
- [4] Jürgen Bokowski and Anselm Eggert, *All realizations of Möbius’ torus with seven vertices*, Structural Topology (1991), no. 17, 59–78.
- [5] Jürgen Bokowski and Antônio Guedes de Oliveira, *On the generation of oriented matroids*, Discrete Comput. Geom. **24** (2000), no. 2-3, 197–208. The Branko Grünbaum birthday issue.
- [6] Jürgen Bokowski and Bernd Sturmfels, *Computational synthetic geometry*, Lecture Notes in Mathematics, vol. 1355, Springer-Verlag, Berlin, 1989.
- [7] Ulrich Brehm, *How to build minimal polyhedral models of the Boy surface*, Math. Intelligencer **12** (1990), no. 4, 51–56.
- [8] David Bremner, *MPC – Matroid Polytope Completion*, June 2004. available at <http://www.cs.unb.ca/profs/bremner/mpc/>.
- [9] Paula Carvalho and Antônio Guedes de Oliveira, *Intersection and linking numbers in oriented matroids*, Discrete Comput. Geom. **31** (2004), no. 2, 305–321.
- [10] Davide P. Cervone, *Vertex-minimal simplicial immersions of the Klein bottle in three space*, Geom. Dedicata **50** (1994), no. 2, 117–141.
- [11] Ákos Császár, *A polyhedron without diagonals*, Acta Univ. Szeged. Sect. Sci. Math. **13** (1949), 140–142.
- [12] Niklas Eén and Niklas Sörensson, *An extensible SAT-solver*, SAT (2003) (Enrico Giunchiglia and Armando Tacchella, eds.), Lecture Notes in Computer Science, vol. 2919, Springer, 2004, pp. 502–518.
- [13] Lukas Finschi and Komei Fukuda, *Generation of oriented matroids—a graph theoretical approach*, Discrete Comput. Geom. **27** (2002), no. 1, 117–136.

- [14] Evgenij Gawrilow and Michael Joswig, *polymake: a Framework for Analyzing Convex Polytopes*, Polytopes — Combinatorics and Computation (Gil Kalai and Günter M. Ziegler, eds.), Birkhäuser, 2000, pp. 43–74.
- [15] Branko Grünbaum, *Convex polytopes*, 2nd ed., Graduate Texts in Mathematics, vol. 221, Springer-Verlag, New York, 2003. Prepared and with a preface by Volker Kaibel, Victor Klee and Günter M. Ziegler.
- [16] Ralf Gugisch, *Konstruktion von Isomorphieklassen orientierter Matroide*, Dissertation, Universität Bayreuth, 2005 (German).
- [17] Stefan Hougardy, Frank H. Lutz, and Mariano Zelke, *Surface realization with the intersection edge functional*, Exp. Math., to appear, available at [arXiv:math.MG/0608538](https://arxiv.org/abs/math/0608538).
- [18] Mark Jungerman and Gerhard Ringel, *Minimal triangulations on orientable surfaces*, Acta Math. **145** (1980), no. 1-2, 121–154.
- [19] Frank H. Lutz, *Enumeration and random realization of triangulated surfaces*, available at [arXiv:math.CO/0506316](https://arxiv.org/abs/math/0506316).
- [20] ———, *Triangulated manifolds with few vertices and vertex-transitive group actions*, Shaker, Aachen, 1999 (English, with German summary). Dissertation, Technische Universität Berlin, Berlin, 1999.
- [21] Thom Sulanke and Frank H. Lutz, *Isomorphism free lexicographic enumeration of triangulated surfaces and 3-manifolds*, Eur. J. Comb., to appear, available at [arXiv:math.CO/0610022](https://arxiv.org/abs/math/0610022).
- [22] Yogesh S. Mahajan, Zhaohui Fu, and Sharad Malik, *Zchaff2004: An efficient SAT solver.*, SAT (2004) (Holger H. Hoos and David G. Mitchell, eds.), Lecture Notes in Computer Science, vol. 3542, Springer, 2005, pp. 360–375.
- [23] Peter McMullen, Christoph Schulz, and Jörg M. Wills, *Polyhedral 2-manifolds in  $E^3$  with unusually large genus*, Israel J. Math. **46** (1983), no. 1-2, 127–144.
- [24] Isabella Novik, *A note on geometric embeddings of simplicial complexes in a Euclidean space*, Discrete Comput. Geom. **23** (2000), no. 2, 293–302.
- [25] Charles S. Peirce, *On the algebra of logic*, Amer. J. Math. **3** (1880), 15–58.
- [26] Simon L. Peyton Jones (ed.), *Haskell 98 language and libraries: the revised report*, Cambridge University Press, Cambridge, 2003.
- [27] Gerhard Ringel, *Map color theorem*, Springer-Verlag, New York, 1974. Die Grundlehren der mathematischen Wissenschaften, Band 209.
- [28] Ernst Steinitz and Hans Rademacher, *Vorlesungen über die Theorie der Polyeder*, Springer, Berlin, 1934.
- [29] Dagmar Timmreck, *Necessary Conditions for Geometric Realizability of Simplicial Complexes*, available at [arXiv:0705.1912](https://arxiv.org/abs/0705.1912).
- [30] Günter M. Ziegler, *Polyhedral surfaces of high genus*, available at [arXiv:math.MG/0412093](https://arxiv.org/abs/math/0412093).

LARS SCHEWE, FACHBEREICH MATHEMATIK, AG OPTIMIERUNG, TU DARMSTADT, SCHLOSSGARTENSTRASSE 7, 64289 DARMSTADT, GERMANY  
*E-mail address:* [schewe@mathematik.tu-darmstadt.de](mailto:schewe@mathematik.tu-darmstadt.de)